Stratification and shear vertical modes in the quasigeostrophic formalism

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Abstract

This is a brief documentation for the matlab routines computing the quasi-geostrophic waves solution for given vertical profiles of density (or Brünt-Väisäila frequency) and geostrophic mean velocity field (which must be plane).

1 Introduction

The following equations and notations follow closely Beckmann (1988) and Laurent Dubus PhD thesis (1999). The streamfunction is related to the perturbation pressure p' by $\psi = p'/(\rho_0 f_0)$ (TO CHECK). The basic equations are the quasigeostrophic vorticity equation:

$$\nabla^2\psi_t + J(\psi, \nabla^2\psi) + \beta\psi_x - f_0 w_z = \text{PVforcing} + \text{PVdissipation}, \tag{1}$$

and the quasigeostrophic density equation ($\rho \equiv \partial_z\psi$):

$$\psi_{zt} + J(\psi, \psi_z) + (N^2/f_0)w = -\gamma H(z - z_m)\psi_z + K\nabla^2\psi_z. \tag{2}$$

The linearized density equation in quasigeostrophy is $D_t\rho' + w\, d_z\overline{\rho} = 0$ with $D_t = \partial_t + \mathbf{u_G}\cdot\nabla$, the geostrophic material derivative (Alain). H is the Heaviside function worth 1 for positive argument, 0 otherwise (0.5 in between?). Note that the surface restoring coefficient γ implicitly takes into account the depth of the upper layer z_m.

Inserting w from (2) in (1), we obtain a single equation:

$$\{\partial_t \cdot + J(\psi, \cdot)\}\left[\nabla^2\psi + \partial_z\left(\frac{f_0^2}{N^2}\partial_z\psi\right)\right] + \beta\partial_x\psi = \partial_z\left[\frac{f_0^2}{N^2}\left(-\gamma H(z - z_m)\psi_z + K\nabla^2\psi_z\right)\right] + \text{PVforcing} + \text{PVdissipation}.$$

The boundary conditions are those of no normal flow, hence in the absence of bottom topography, $w = 0$ at the surface ($z = 0$) and the bottom ($z = -H$):

$$\psi_{zt} + J(\psi, \psi_z) = -\gamma H(z - z_m)\psi_z + K\nabla^2\psi_z.$$

We assume a robust decomposition of the flow into a mean Ψ (in thermal wind balance) and a plane wave perturbation ψ':

$$\psi = \Psi + \psi',$$

for instance $\Psi = 0$ and $\psi' = F(z)\ \exp[i(kx+ly-\omega t)]$. In the subsequent derivations of the equations for the perturbation given a specified mean flow, we keep only the linear terms.

2 Stratification modes

Here we want to estimate the preferred vertical structure of plane waves solutions for a given stratification but no mean flow ($\Psi \equiv 0$). Because of the separability of the problem, we can consider a slightly more general form of the perturbation:

$$\psi' = F(z)\ M(y)\ \exp[i(kx - \omega t)].$$

Keeping only the linear terms in the vorticity and density equations results in the well-known dispersion relation for baroclinic Rossby waves:

$$\omega[(k^2 + \lambda^2)M - M_{yy}] + \beta k M = 0,$$

and the Sturm-Liouville problem for the vertical dependence:

$$[(f_0/N)^2 F_z]_z + \lambda^2 F = 0,$$

with $F_z = 0$ at the top ($z = 0$) and the bottom ($z = -H$). The separation parameter is the inverse internal Rossby radius of deformation: $\lambda = 1/R_i$.

The matlab routine follows:

```
% Compute baroclinic Rossby radii of deformation [m].
% Input: nsize, dz(1,nsize), N2(1,nsize) only(2:nsize)used  f0
% Output: Modes(nsize,nsize) Rossby(1,nsize)/km
  Dzarray  = dz(1:nsize)'; % [m] (1,nsize)
  Dzzarray = 0.5*([ Dzarray 0 ] + [ 0 Dzarray ]); % [m] (1,nsize+1)
  FON2=(f0*f0)./N2(2:nsize)./Dzzarray(2:nsize);
  V = -([0 FON2]+[FON2 0])./Dzarray;
  Vup   = FON2./Dzarray(1:nsize-1);
  Vdown = FON2./Dzarray(2:nsize);
  MAT_VMODES=diag(V)+diag(Vup,1)+diag(Vdown,-1); % (nsize,nsize) [(f/N)^2 Vz]z
  [Modes,D]=eig(MAT_VMODES);
  [Lambda,k]=sort(diag(D));
  Modes=Modes(:,k);
  Rossby=1.0e-3./sqrt(abs(Lambda)); % [km]
```

NB Matrix MAT_VMODES multiplied by a vector V realizes the derivation $[(f_0/N)^2 V_z]_z$, assuming $V_z = 0$ at the top and the bottom: This is **not** suitable for the barotropic mode, that requires a free-surface boundary condition! (cf notes vertical_modes for the correct derivation).

Matrix MAT_VMODES is symmetric if dz is constant, such that the derivation operator is self-adjoint: its eigenvectors compose an orthonormal base and its eigenvalues are real. If dz is not constant, MAT_VMODES is no longer symmetric, and the scalar product must be weighted by dz for the eigenvectors to be orthogonal:

$$< u, v >= H^{-1} \int_{-H}^{0} u\, v\, dz.$$

See notes vertical_modes for the simple proof of orthogonality.

N = constant

In the special case where the stratification is uniform, ie N does not depend on z, analytical solutions can be found:

$$\partial_{zz} F = -\lambda^2 \frac{N_0^2}{f_0^2} F = -\sigma^2 F.$$

The general solutions are of the form $F = A \sin \sigma z + B \cos \sigma z$ and must satisfy the boundary conditions, implying $A = 0$ $(z = 0)$ and $\sigma = n\pi/H$ $(z = -H)$, hence $\lambda_n = n\pi f_0/(HN)$, or $R_n = NH/(n\pi f_0)$, $\omega_n = -\beta k/(k^2 + l^2 + \lambda_n^2)$, and $c_n = \omega_n/k = -\beta/(k^2 + l^2 + \lambda_n^2)$. In the long wave limit, the phase and group velocity is $c_n = -\beta R_n^2 = -\beta N^2 H^2/(n^2 \pi^2 f_0^2)$. The lower baroclinic vertical mode propagates westward 4 times faster than the second!

3 Shear modes

The mean flow has now the more general structure of a horizontal velocity field depending on depth (assumed in thermal wind balance):

$$\Psi = V(z)\ x - U(z)\ y.$$

We obtain the following equation for the perturbations:

$$\begin{aligned}\{\partial_t \cdot + U\partial_x \cdot + V\partial_y \cdot\} \left[\nabla^2\psi + \partial_z \left(\frac{f_0^2}{N^2}\partial_z\psi\right)\right] + \left[\beta - \partial_z \left(\frac{f_0^2}{N^2} d_z U\right)\right] \partial_x\psi - \left[\partial_z \left(\frac{f_0^2}{N^2} d_z V\right)\right] \partial_y\psi \\ = A\,\nabla^4\psi - A_4\,\nabla^6\psi + K\,\nabla^2 \left[\partial_z \left(\frac{f_0^2}{N^2}\partial_z\psi\right)\right] - \gamma\,\partial_z \left(\frac{f_0^2}{N^2} H(z - z_m)\partial_z\psi\right)\end{aligned}$$

with the boundary conditions at $z = 0, -H$:

$$\{\partial_t \cdot + U\partial_x \cdot + V\partial_y \cdot\}\, \partial_z\psi - d_z V\, \partial_y\psi - d_z U\, \partial_x\psi = -\gamma\, H(z - z_m)\partial_z\psi + K\nabla^2\partial_z\psi.$$

We assume the perturbation is a plane wave with vertical structure F:

$$\psi' = F(z)\ \exp[i(kx + ly - \omega t)].$$

We also add some traditional horizontal dissipation operators, namely a Laplacian density diffusion (K in [m^2/s]) and surface restoring (γ, in s^{-1}), Laplacian (A in [m^2/s]) and biharmonic ($A_4 > 0$ in [m^4/s]) viscosity, and bottom friction.

The generalized eigenvalue problem one obtains after linearizing the vorticity and density equations reads:

$$\omega\{(k^2+l^2)F - [(f_0/N)^2 F_z]_z\} = -\beta k F + [(f_0/N)^2 U_z]_z kF + [(f_0/N)^2 V_z]_z lF$$
$$+(kU+lV)\{(k^2+l^2)F - [(f_0/N)^2 F_z]_z\}$$
$$-i[A(k^4+l^4) + A_4(k^6+l^6)]F + iK(k^2+l^2)[(f_0/N)^2 F_z]_z + i\gamma\,\partial_z\left(\frac{f_0^2}{N^2}H(z-z_m)\partial_z F\right).$$

I first assumed incorrect boundary conditions $F_z = U_z = V_z = 0$ at the top and the bottom. But in fact the correct boundary conditions at $z = 0, -H$ are no normal flow (Smith 2007, Hochet et al. 2015):

$$(kU + lV - \omega)\partial_z F = (k\partial_z U + l\partial_z V)F + i\,\gamma\,H(z-z_m)\partial_z F + i\,K\,(k^2+l^2)\partial_z F.$$

However, apparently (see Appendix), their implementation in the full equations for the upper and lower layers ends up in the cancellation of all the boundary terms, leaving the equations unchanged from the wrong previous simpler boundary conditions.
NB For the perturbation equation to contain no zeroth order term, we implicitly assume that the forcing terms are in balance with the mean flow, i.e. $\beta V = Vforcing$ and $UV_z - VU_z = Dforcing$. These equations were derived by Pedlosky (1987) in polar coordinates for analytical purpose, however the implementation of the code from a purely zonal flow formulation (A. M. Treguier, modified by L. Dubus) is simpler in the (k,l) form.

In the following matlab code, the eigenvalue problem is written using matrix operators:

$$\omega \mathrm{MK2L2}F = \mathrm{MAT\_RHS}F = (\mathrm{MAT\_BETA} + \mathrm{MAT\_ADVECT} + \mathrm{MAT\_DISSIP})F.$$

```
% Compute speed, growth rate and vertical structure of QG shear modes.
% Input: nsize dz(1,nsize) N2(1,nsize) u(nsize,1) v(nsize,1)
% wavenumbers(kx,ly) f0
% nubih nulap nubot
% nudif = 0.0;    % [m^2/s] horizontal diffusion of density
% Output: Umodes, omega_complex
% kx (ly) is the zonal (meridional) wavenumber, scalar or vector [m-1]
  numk = max(size(kx)); numl = max(size(ly)) ;
  most_mode_1   = zeros(nsize,numk,numl);
  most_growth_1 = zeros(numk,numl);
  most_speed_1  = zeros(numk,numl);
  Dzarray  = dz(1:nsize)'; % [m] (1,nsize)
  Dzzarray = 0.5*([ Dzarray 0 ] + [ 0 Dzarray ]); % [m] (1,nsize+1)
  F0N2=(f0*f0)./N2(2:nsize)./Dzzarray(2:nsize);
  V=-([0 F0N2]+[F0N2 0])./Dzarray;
```

```
  Vup   = F0N2./Dzarray(1:nsize-1);
  Vdown = F0N2./Dzarray(2:nsize  );
  MAT_VMODES=diag(V)+diag(Vup,1)+diag(Vdown,-1);
  d2udz=MAT_VMODES*u; d2vdz=MAT_VMODES*v;
  clear i; % i must be the complex root of (-1)
  for il=1:numl % loop over meridional wavenumber
   for ik=1:numk % loop over zonal      wavenumber
    k2l2 = -(kx(ik)^2+ly(il)^2); k4l4 = k2l2*k2l2; k6l6 = k4l4*k2l2;
    dissip = i*(nulap*k4l4-nubih*k6l6);
%   vector for bottom friction as a body force over the last layer:
    bot_fric = zeros(1,nsize); bot_fric(nsize)=-i*k2l2*nubot;
%   forms matrix multiplying omega: -(k2+l2)+d/dz(f02/N2 d/dz
    MK2L2 = eye(nsize)*k2l2+MAT_VMODES;
%   The rhs matrix is made of beta effect, advection and dissipation.
    MAT_BETA   = beta*kx(ik)*eye(nsize)-kx(ik)*diag(d2udz)-ly(il)*diag(d2vdz);
    MAT_ADVECT = (diag(u)*kx(ik)+diag(v)*ly(il))*MK2L2;
    MAT_DISSIP = dissip*eye(nsize) + diag(bot_fric) + i*nudif*k2l2*MAT_VMODES;
    MAT_RHS    = MAT_BETA + MAT_ADVECT + MAT_DISSIP;
    [Umodes,D] = eig(MAT_RHS,MK2L2);
%   eig output: diagonal matrix D containing complex eigenvalues.
    Vi = imag(diag(D)); Vr = real(diag(D));
    if max(Vi) > critical_growth       % when there are positive growth rates,
%     sort growth rates in ascending order, then omega and unstable modes.
      [growth,indexk]=sort(Vi);
      Umodes=Umodes(:,indexk); speed=Vr(indexk);
      most_mode_1(:,ik,il) = Umodes(:,nsize);
      most_growth_1(ik,il) = growth(nsize);
      most_speed_1(ik,il)  = speed(nsize);
    end
   end % end loop over zonal      wavenumber
  end % end loop over meridional wavenumber
```

NB One could alternatively look for the eigenvalues of (MK2L2^{-1} MAT_RHS) instead of the generalized eigenvalue problem. Because of the advection with sheared velocities, MAT_ADVECT hence MAT_RHS and (MK2L2^{-1} MAT_RHS) are not symmetric (for uniform dz, MK2L2 and its inverse are symmetric): the sheared modes are not orthogonal, and thus allow non-normal growth (a concept much appreciated by Florian Sévellec).

Appendix: approximate implementation of the correct boundary conditions

With $dz(1:n)$ the layer thicknesses, the vorticity conservation equation for the upper layer reads:

$$\omega\left\{(k^2+l^2)F-\frac{(f_0/N)^2F_z|_{z=0}-(f_0/N)^2F_z|_{z=-dz(1)}}{dz(1)}\right\}=-\beta kF$$
$$+\frac{(f_0/N)^2U_z|_{z=0}-(f_0/N)^2U_z|_{z=-dz(1)}}{dz(1)}kF+\frac{(f_0/N)^2V_z|_{z=0}-(f_0/N)^2V_z|_{z=-dz(1)}}{dz(1)}lF$$
$$+(kU+lV)\left\{(k^2+l^2)F-\frac{(f_0/N)^2F_z|_{z=0}-(f_0/N)^2F_z|_{z=-dz(1)}}{dz(1)}\right\}$$
$$-i[A(k^4+l^4)+A_4(k^6+l^6)]F+iK(k^2+l^2)\frac{(f_0/N)^2F_z|_{z=0}-(f_0/N)^2F_z|_{z=-dz(1)}}{dz(1)}$$
$$+i\gamma\,\frac{(f_0/N)^2H(z-z_m)\partial_zF|_{z=0}-(f_0/N)^2H(z-z_m)\partial_zF|_{z=-dz(1)}}{dz(1)}.$$

The terms involving vertical derivatives at the surface (in red) cancel altogether because of the upper boundary condition (except eventually for the eddy diffusivity), assuming $F(z=0)\approx F(z=-0.5dz(1))$, $U(z=0)\approx U(z=-0.5dz(1))$, and $V(z=0)\approx V(z=-0.5dz(1))$. This is very convenient, especially since a more precise implementation requires additional hypothesis! The same applies for the lower layer equation under the same approximations:

$$\omega\left\{(k^2+l^2)F-\frac{(f_0/N)^2F_z|_{z=-H+dz(n)}-(f_0/N)^2F_z|_{z=-H}}{dz(n)}\right\}=-\beta kF$$
$$+\frac{(f_0/N)^2U_z|_{z=-H+dz(n)}-(f_0/N)^2U_z|_{z=-H}}{dz(n)}kF+\frac{(f_0/N)^2V_z|_{z=-H+dz(n)}-(f_0/N)^2V_z|_{z=-H}}{dz(n)}lF$$
$$+(kU+lV)\left\{(k^2+l^2)F-\frac{(f_0/N)^2F_z|_{z=-H+dz(n)}-(f_0/N)^2F_z|_{z=-H}}{dz(n)}\right\}$$
$$-i[A(k^4+l^4)+A_4(k^6+l^6)]F+iK(k^2+l^2)\frac{(f_0/N)^2F_z|_{z=-H+dz(n)}-(f_0/N)^2F_z|_{z=-H}}{dz(n)}.$$

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